An elementary proof of a distribution relation on elliptic curves*

Frazer Jarvis
Department of Pure Mathematics
University of Sheffield
Sheffield S3 7RH
U.K.
a.f.jarvis@shef.ac.uk

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Abstract

We give another elementary proof of a certain identity of elliptic functions arising from the K-theory of elliptic curves and Wildeshaus's generalisation of Zagier's conjectures. This proof consists of a calculation with the q-expansions, and is offered in the hope that its more explicit flavour may be generalised to other situations.

Introduction

Fix $\mathbb{Z}$-lattices L, $L' \subset \mathbb{C}$ such that there is an isogeny $\psi : \mathbb{C}/L \longrightarrow \mathbb{C}/L'$.

We recall the well-known Siegel function,

$$\varphi_L(z) = \varphi(z, L) = e^{-z\eta(z,L)/2}\sigma(z, L)\Delta(L)^{1/12},$$

defined for $z \in \mathbb{C}\backslash L$, and extend its definition linearly to divisors $D = \sum_{i=1}^{r} a_i(z_i)$ in $\mathbb{C}$ by

$$\varphi_L(D) = \prod_{\substack{i=1 \\ z_i \notin L}}^{r} \varphi_L(z_i)^{a_i}.$$

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We will be particularly interested in divisors of the form

$$(x, y) = (x + y) + (x - y) - 2(x) - 2(y) + 2(0)$$

which have the property that they lie in (and, in some sense, generate) the kernel of the squaring homomorphism

$$q_2 : \mathrm{Pic}^0(\mathbb{C}) \longrightarrow \mathrm{Sym}^2(\mathbb{C})$$

defined by $q_2((z) - (0)) = z \otimes z$ and extending linearly. We explained in [1] that, given a divisor D on $\mathbb{C}/L$ in the kernel of this squaring map q_2, we define $\varphi_L(D)$ to be $\varphi_L(\widetilde{D})$, where $\widetilde{D} = \sum_{i=1}^{r} a_i(z_i)$ is a divisor on $\mathbb{C}$ such that $\sum_{i=1}^{r} a_i z_i \otimes z_i = 0$ in $\mathbb{C} \otimes \mathbb{C}$. I am grateful to Norbert Schappacher for pointing out the remark (due to Jörg Wildeshaus) that this is the correct choice of lift of D to $\mathbb{C}$ in this situation.

Then we prove the following:

Theorem 0.1 *Suppose that x, $y \in \mathbb{C}/L$ are such that x, y, $x + y$ and $x - y$ do not lie in $\ker \psi$. Write D for the divisor*

$$(x, y) = (x + y) + (x - y) - 2(x) - 2(y) + 2(0)$$

in $\mathbb{C}/L$. Then

$$\varphi_{L'}(D) = \prod_{w \in \ker \psi} \varphi_L(D \oplus w)$$

where $D \oplus w$ denotes the translate of the divisor D obtained by adding w to every point in its support.

Throughout the paper, the symbol $=$ will indicate equality of two complex numbers only up to a root of unity; we are therefore really working in $\mathbb{C}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$. This implies that quantites such as $\Delta(L)^{\frac{1}{12}}$ above are well-defined. The reader will readily determine the power to which to raise each expression in order to get genuine equality, but we prefer not to complicate the notation too much.

The importance of the result is explained in [2] and [3]. The reader is referred to these sources and to the original papers of Wildeshaus ([5] and [6]) for a discussion of Zagier's conjecture in the setting of elliptic curves.

1 The proof

We first consider the behaviour of φ under isomorphisms.

Lemma 1.1 *For $\alpha \in \mathbb{C}^\times$, suppose that the isogeny*

$$\begin{array}{rcl} \psi : \mathbb{C}/L & \twoheadrightarrow & \mathbb{C}/L' \\ z & \mapsto & \alpha z \end{array}$$

is an isomorphism—i.e., $L' = \alpha L$. Then

$$\varphi(\alpha z, L') = \varphi(z, L).$$

Proof. One has

$$\begin{array}{rcl} \eta(\alpha z, \alpha L) & = & \alpha^{-1}\eta(z, L) \\ \sigma(\alpha z, \alpha L) & = & \alpha\sigma(z, L) \\ \Delta(\alpha L) & = & \alpha^{-12}\Delta(L). \end{array}$$

The result immediately follows.

In particular, one may always reduce to the case $L = \mathbb{Z} + \mathbb{Z}\tau$, with τ in the upper half-complex plane, by scaling a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ to $\mathbb{Z} + \mathbb{Z}(\omega_2/\omega_1)$, and replacing ω_2 by $-\omega_2$ if necessary to ensure that $\mathrm{im}(\omega_2/\omega_1) > 0$.

In the case that L is of the particular form

$$\begin{array}{rcl} L & = & \mathbb{Z}\tau + \mathbb{Z} \\ z & = & a\tau + b \end{array}$$

we can give a q-expansion (i.e., Fourier expansion) for φ_L.

One knows that

$$\sigma(z, L) = -\frac{1}{2\pi i} e^{\eta(1,L)z^2/2} e^{-\pi i z}(1 - u) \prod_{n \geq 1} \frac{(1 - q^n u)(1 - q^n u^{-1})}{(1 - q^n)^2}$$

(where, as usual, $u = e^{2\pi i z}$, $q = e^{2\pi i \tau}$). Also,

$$\Delta(L) = (2\pi i)^{12} q \prod_{n \geq 1} (1 - q^n)^{24}.$$

The Legendre relation is:

$$e^{\eta(1)z^2/2 - z\eta(z)/2} = e^{z[z\eta(1) - \eta(z)]/2} = e^{z\pi i a}.$$

It follows that:

$$\begin{array}{rcl} \varphi(z, L) & = & e^{\pi i a z} e^{-\pi i z} q^{1/12}(1 - u) \prod_{n \geq 1} (1 - q^n u)(1 - q^n u^{-1}) \\ & = & q^{\frac{1}{2}a^2 - \frac{1}{2}a + \frac{1}{12}} e^{2\pi i b(a-1)/2}(1 - u) \prod_{n \geq 1} (1 - q^n u)(1 - q^n u^{-1}). \end{array}$$

Remark 1.2 The second Bernoulli polynomial, $B_2(X)$, is equal to $X^2 - X + \frac{1}{6}$. It is not a surprise that this should appear in this setting; Bernoulli polynomials appear throughout the theory—see [4], for example, where the third Bernoulli polynomial plays an important role.

Now one considers a more general isogeny. Such a function

$$\begin{array}{rcl} \psi : \mathbb{C}/L & \twoheadrightarrow & \mathbb{C}/L' \\ z & \mapsto & \alpha z \end{array}$$

factorises as

$$\psi : \mathbb{C}/L \xrightarrow{\sim} \mathbb{C}/\alpha L \twoheadrightarrow \mathbb{C}/L',$$

where the second morphism is the natural projection. It remains to prove the result for projections

$$\begin{array}{rcl} \psi : \mathbb{C}/L & \twoheadrightarrow & \mathbb{C}/L' \\ z & \mapsto & z \end{array}$$

where $L' \supset L$.

By the theory of elementary divisors, there exists a basis $\{\omega_1, \omega_2\}$ of L' such that

$$\begin{array}{rcl} L' & = & \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \\ L & = & (A\mathbb{Z})\omega_1 + (B\mathbb{Z})\omega_2 \end{array}$$

where $B|A$. Put $N = A/B$. Then the projection ψ further factorises as

$$\psi : \mathbb{C}/L \twoheadrightarrow \mathbb{C}/L'' \twoheadrightarrow \mathbb{C}/L',$$

where

$$L'' := (N\mathbb{Z})\omega_1 + \mathbb{Z}\omega_2.$$

It thus suffices to prove the result in the two cases where either

$$\begin{array}{rcl} L' & = & \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \\ L & = & (N\mathbb{Z})\omega_1 + \mathbb{Z}\omega_2, \end{array}$$

or

$$\begin{array}{rcl} L' & = & \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \\ L & = & (B\mathbb{Z})\omega_1 + (B\mathbb{Z})\omega_2. \end{array}$$

Recall the definition of the symbol (x, y). Here, x and y are points of $\mathbb{C}/L$, which lift to points z, $w \in \mathbb{C}$. We give the q-expansion for $\varphi((x, y))$.

Let $L = \mathbb{Z}\tau + \mathbb{Z}$. Put

$$\begin{aligned} q &= e^{2\pi i\tau} \\ u &= e^{2\pi iz} \\ v &= e^{2\pi iw}. \end{aligned}$$

Then one has

$$\varphi((x,y),L) = \frac{\sigma(z+w,L)\sigma(z-w,L)}{\sigma(z,L)^2\sigma(w,L)^2\Delta(L)^{\frac{1}{6}}} =$$

$$q^{-\frac{1}{6}}\frac{(1-uv)(v-u)}{(1-u)^2(1-v)^2}\prod_{n\geq 1}\frac{(1-q^nuv)(1-q^nu^{-1}v^{-1})(1-q^nuv^{-1})(1-q^nu^{-1}v)}{(1-q^nu)^2(1-q^nu^{-1})^2(1-q^nv)^2(1-q^nv^{-1})^2}.$$

(This follows from the q-expansion of σ.)

Write $((x,y)\oplus t)$ for $(x+y+t)+(x-y+t)-2(x+t)-2(y+t)$ (and, more generally, if X is any divisor, we will write $(X\oplus t)$ for the same divisor, but adding t to each point).

Lemma 1.3 *Let $\psi : \mathbb{C}/L \twoheadrightarrow \mathbb{C}/L'$ be the natural projection as above. Then*

$$\sum_{t_1,t_2\in\ker\psi}(x+t_1,y+t_2) = \deg\psi.\sum_{t\in\ker\psi}((x,y)\oplus t),$$

assuming that no term in the support of the left-hand side is 0.

Proof. One expands the left-hand side.

First consider the case where

$$\begin{aligned} L' &= \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \\ L &= (N\mathbb{Z})\omega_1 + \mathbb{Z}\omega_2. \end{aligned}$$

Then

$$\ker\psi = \{m\omega_1 | m \in \mathbb{Z}/N\mathbb{Z}\}.$$

Let $\tau = \omega_2/\omega_1$, $q = e^{2\pi i\tau}$. Also let

$$\begin{aligned} L_0' &= L'/\omega_1 = \mathbb{Z} + \mathbb{Z}\tau \\ L_0 &= L/N\omega_1 = \mathbb{Z} + \mathbb{Z}(\frac{\tau}{N}). \end{aligned}$$

If $z = b\omega_1 + a\omega_2$, $w = d\omega_1 + c\omega_2$, put

$$\begin{aligned} Z = z/\omega_1 &= a\tau + b = (Na)(\frac{\tau}{N}) + b \\ W = w/\omega_1 &= c\tau + d = (Nc)(\frac{\tau}{N}) + d \end{aligned}$$

and $u = e^{2\pi iZ}$, $v = e^{2\pi iW}$. Let $t_i = m_i\omega_1 \in \ker\psi$. Then

Proposition 1.4 *One has*

$$\prod_{t\in\ker\psi}\varphi(((x,y)\oplus t),L)=\varphi((x,y),L').$$

Proof. One has

$$[\prod_{t\in\ker\psi}\varphi(((x,y)\oplus t),L)]^N$$

$$\begin{aligned}
&= \prod_{t_1,t_2\in\ker\psi}\varphi((x+t_1,y+t_2),L)\\
&= \prod_{m_1,m_2=0}^{N-1}\varphi((\frac{Z+m_1}{N},\frac{W+m_2}{N}),L_0) \qquad \text{(using Lemma 1.2)}\\
&= \prod_{m_1,m_2=0}^{N-1}\left[(q^{\frac{1}{N}})^{-\frac{1}{6}}\frac{(1-e^{\frac{2\pi i}{N}(Z+m_1+W+m_2)})(e^{\frac{2\pi i}{N}(W+m_2)}-e^{\frac{2\pi i}{N}(Z+m_1)})}{(1-e^{\frac{2\pi i}{N}(Z+m_1)})^2(1-e^{\frac{2\pi i}{N}(W+m_2)})^2}\right]\cdot\\
&\quad \prod_{m_1,m_2=0}^{N-1}\prod_{n\geq1}\frac{(1-q^{\frac{n}{N}}e^{\frac{2\pi i}{N}(Z+m_1+W+m_2)})(1-q^{\frac{n}{N}}e^{\frac{2\pi i}{N}(Z-W+m_1-m_2)})}{(1-q^{\frac{n}{N}}e^{\frac{2\pi i}{N}(Z+m_1)})^2(1-q^{\frac{n}{N}}e^{-\frac{2\pi i}{N}(Z+m_1)})^2}\cdot\\
&\quad \frac{(1-q^{\frac{n}{N}}e^{\frac{2\pi i}{N}(-Z-W-m_1-m_2)})(1-q^{\frac{n}{N}}e^{\frac{2\pi i}{N}(-Z+W-m_1+m_2)})}{(1-q^{\frac{n}{N}}e^{\frac{2\pi i}{N}(W+m_2)})^2(1-q^{\frac{n}{N}}e^{-\frac{2\pi i}{N}(W+m_2)})^2}
\end{aligned}$$

When m_i run over the elements $0,\ldots,N-1$, the values of $e^{\frac{2\pi i}{N}m_1}$, $e^{\frac{2\pi i}{N}m_2}$, $e^{\frac{2\pi i}{N}(m_1+m_2)}$, and $e^{\frac{2\pi i}{N}(m_1-m_2)}$ run through the set of Nth roots of unity N times each. From the equation

$$(1-x^N)=\prod_{m=0}^{N-1}(1-\zeta_N^m x),$$

one finds

$$[\prod_{t\in\ker\psi}\varphi(((x,y)\oplus t),L)]^N$$

$$\begin{aligned}
&= \left[q^{-\frac{1}{6}}\frac{(1-uv)(v-u)}{(1-u)^2(1-v)^2}\right]^N\\
&\quad \left[\prod_{n\geq1}\frac{(1-q^nuv)(1-q^nu^{-1}v^{-1})(1-q^nuv^{-1})(1-q^nu^{-1}v)}{(1-q^nu)^2(1-q^nu^{-1})^2(1-q^nv)^2(1-q^nv^{-1})^2}\right]^N\\
&=\varphi((x,y)/\omega_1,L_0')^N\\
&=\varphi((x,y),L')^N,
\end{aligned}$$

(again using Lemma 1.2) as desired.

Now one considers the second case. It is more complicated, but also elementary.

Suppose that

$$\begin{aligned} L' &= \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \\ L &= (B\mathbb{Z})\omega_1 + (B\mathbb{Z})\omega_2. \end{aligned}$$

Then

$$\ker\psi = \{s\omega_1 + t\omega_2 | s, t \in \mathbb{Z}/B\mathbb{Z}\}.$$

Let $\tau = \omega_2/\omega_1$, $q = e^{2\pi i\tau}$. Let

$$L_0 = \mathbb{Z} + \mathbb{Z}\tau.$$

If $z = b\omega_1 + a\omega_2$, $w = d\omega_1 + c\omega_2$, put

$$\begin{aligned} Z = z/\omega_1 &= a\tau + b \\ W = w/\omega_1 &= c\tau + d \end{aligned}$$

and $u = e^{2\pi i Z}$, $v = e^{2\pi i W}$. Then

Proposition 1.5 *One has*

$$\prod_{t\in\ker\psi} \varphi(((x,y)\oplus t), L) = \varphi((x,y), L').$$

Proof. One has

$$[\prod_{t\in\ker\psi} \varphi(((x,y)\oplus t), L)]^{B^2}$$

$$\begin{aligned}
&= \prod_{t_1,t_2\in\ker\psi} \varphi((x+t_1, y+t_2), L) \\
&= \prod_{r_1,s_1,r_2,s_2=0}^{B-1} \varphi((\frac{Z+r_1\tau+s_1}{B}, \frac{W+r_2\tau+s_2}{B}), L_0) \\
&= \prod_{r_i,s_i} q^{-\frac{1}{6}} \frac{(1-e^{\frac{2\pi i}{B}(Z+W+(r_1+r_2)\tau+s_1+s_2)})(e^{\frac{2\pi i}{B}(W+r_2\tau+s_2)} - e^{\frac{2\pi i}{B}(Z+r_1\tau+s_1)})}{(1-e^{\frac{2\pi i}{B}(Z+r_1\tau+s_1)})^2(1-e^{\frac{2\pi i}{B}(W+r_2\tau+s_2)})^2}. \\
&\quad \prod_{r_i,s_i}\prod_{n\geq 1}\cdots
\end{aligned}$$

First take the product over s_1 and s_2:

$$\begin{aligned}
= \quad & q^{-\frac{B^4}{6}} \prod_{r_i=0}^{B-1} \frac{(1-q^{r_1+r_2}uv)^B (q^{r_2}v - q^{r_1}u)^B}{(1-q^{r_1}u)^{2B}(1-q^{r_2}v)^{2B}} \\
& \prod_{r_i=0}^{B-1}\prod_{n\geq 1} \frac{(1-q^{Bn+r_1+r_2}uv)^B(1-q^{Bn+r_1-r_2}uv^{-1})^B}{(1-q^{Bn+r_2}u)^{2B}(1-q^{Bn-r_2}u^{-1})^{2B}} \\
& \frac{(1-q^{Bn-r_1-r_2}u^{-1}v^{-1})^B(1-q^{Bn-r_1+r_2}u^{-1}v)^B}{(1-q^{Bn+r_2}v)^{2B}(1-q^{Bn-r_2}v^{-1})^{2B}}
\end{aligned}$$

Note that the denominator of this expression is the same as that for the series for $\varphi((x,y),L')^{B^2}$. This follows as one has products of the following form:

$$\prod_{r_i=0}^{B-1}(1-q^{r_1}u)\prod_{r_i=0}^{B-1}\prod_{n\geq 1}(1-q^{Bn+r_1}u) = (1-u)^B\prod_{n\geq 1}(1-q^n)^B.$$

Thus we concentrate on the numerator of this expression. Note that

$$\begin{aligned}
\prod_{r_i}(q^{r_2}v - q^{r_1}u)^B &= \prod_{r_i}(q^{r_2}v(1-q^{r_1-r_2}uv^{-1}))^B \\
&= q^{B^2\cdot\frac{(B-1)B}{2}}v^{B^3}\prod_{r_i}(1-q^{r_1-r_2}uv^{-1}).
\end{aligned}$$

Then the numerator is:

$$\begin{aligned}
& q^{-\frac{B^4}{6}} q^{\frac{1}{2}B^3(B-1)} v^{B^3}. \\
& [\prod_{a\in\mathbb{Z}}(1-q^a uv)^{k_1(a)}(1-q^a uv^{-1})^{k_2(a)}(1-q^a u^{-1}v^{-1})^{k_3(a)}(1-q^a u^{-1}v)^{k_4(a)}]^B,
\end{aligned}$$

where:

$$\begin{aligned}
k_1(a) &= \begin{cases} 0, & \text{if } a \leq -1, \\ a+1, & \text{if } 0 \leq a \leq B-1, \\ B, & \text{if } B-1 \leq a. \end{cases} \\
k_2(a) &= \begin{cases} 0, & \text{if } a \leq -B, \\ a+B, & \text{if } 1-B \leq a \leq 0, \\ B, & \text{if } 0 \leq a. \end{cases} \\
k_3(a) &= \begin{cases} 0, & \text{if } a \leq 1-B, \\ a+B-1, & \text{if } 2-B \leq a \leq 1, \\ B, & \text{if } 1 \leq a. \end{cases} \\
k_4(a) &= \begin{cases} 0, & \text{if } a \leq 0, \\ a, & \text{if } 1 \leq a \leq B, \\ B, & \text{if } B \leq a. \end{cases}
\end{aligned}$$

One calculates:

$$\frac{\prod_{a\in\mathbb{Z}}(1-q^a uv)^{k_1(a)}(1-q^a uv^{-1})^{k_2(a)}(1-q^a u^{-1}v^{-1})^{k_3(a)}(1-q^a u^{-1}v)^{k_4(a)}}{\prod_{n\geq 1}(1-q^n uv)^B(1-q^n uv^{-1})^B(1-q^n u^{-1}v^{-1})^B(1-q^n u^{-1}v)^B}$$

as

$$\begin{aligned} &(1-uv)(1-quv)^{2-B}(1-q^2uv)^{3-B}\cdots(1-q^{B-2}uv)^{-1}\\ &\quad (1-q^{1-B}uv^{-1})(1-q^{2-B}uv^{-1})^2\cdots(1-q^{-1}uv^{-1})^{B-1}(1-uv^{-1})^B\\ &\quad (1-q^{2-B}u^{-1}v^{-1})(1-q^{3-B}u^{-1}v^{-1})^2\cdots(1-u^{-1}v^{-1})^{B-1}\\ &\quad (1-qu^{-1}v)^{1-B}(1-q^2u^{-1}v)^{2-B}\cdots(1-q^{B-1}u^{-1}v)^{-1}\end{aligned}$$

But one has relations of the form:

$$(1-q^a uv) = (-q^a uv)(1-q^{-a}u^{-1}v^{-1}),$$

so that the product is

$$\begin{aligned} & (1-uv).(\frac{-1}{quv})^{B-2}(\frac{-1}{q^2uv})^{B-3}\cdots(\frac{-1}{q^{B-2}uv}).(-uv)^{1-B}(1-uv)^{B-1}\\ & (1-uv^{-1})^B.(-q^{1-B}uv^{-1})(-q^{2-B}uv^{-1})^2\cdots(-q^{-1}uv^{-1})^{B-1}\\ = \; & (1-uv)^B(1-uv^{-1})^B.v^{1-B}(qv)^{2-B}(q^2v)^{3-B}\cdots(q^{B-2}v)^{-1}\\ & \quad (q^{B-1}v)^{-1}(q^{B-2}v)^{-2}\cdots(qv)^{1-B}\\ = \; & (1-uv)^B(1-uv^{-1})^B.v^{B-B^2}q^{-\frac{1}{6}B(B-1)(2B-1)}\end{aligned}$$

Because

$$-\frac{B^4}{6}+\frac{B^3(B-1)}{2}-\frac{B^2(B-1)(2B-1)}{6} = -\frac{B^2}{6}$$

and

$$B^3 + B(B-B^2) = B^2,$$

it follows that

$$\begin{aligned} & [\prod_{t\in\ker\psi}\varphi(((x,y)\oplus t),L)]^{B^2}\\ = \; & q^{-\frac{B^2}{6}}\left(\frac{(1-uv)(1-uv^{-1})}{(1-u)^2(1-v)^2}\right)^{B^2} v^{B^2}\\ & \left[\prod_{n\geq 1}\frac{(1-q^nuv)(1-q^nu^{-1}v^{-1})(1-q^nuv^{-1})(1-q^nu^{-1}v)}{(1-q^nu)^2(1-q^nu^{-1})^2(1-q^nv)^2(1-q^nv^{-1})^2}\right]^{B^2}\\ = \; & \varphi((x,y)/\omega_1, L_0)^{B^2}\\ = \; & \varphi((x,y), L')^{B^2}.\end{aligned}$$

This concludes the proof.

Acknowledgements

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