13, 31, and the $3x + 1$ problem

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Eureka 49 (March 1989) 22–25 [Graham Nelson, editor]

Problem: Given a positive integer, apply the following algorithm. If it is even, halve it, and if it is odd, multiply it by three and add one. Repeat this process iteratively.

Example: Start with 3. This is odd, so multiply it by three and add one. This gives 10, which is even, so halve this, to obtain 5. Repeated action of this process gives 16, 8, 4, 2, 1, 4, 2, 1,... and we have entered a loop.

Conjecture: Whichever number we start with, the process will always culminate in the loop ..., 4, 2, 1, 4, 2, 1,....

Define: $T(n) =$ the result of the algorithm on n. Define $T^k(n) = T(T^{k-1}(n))$.

The problem dates from before World War II, when it seems to have been proposed by Lothar Collatz, whilst a student at Hamburg [1], but it has resisted all subsequent attempts at solution, despite having been verified up to 10^{12}.

The behaviour of individual numbers of the sequence n, $T(n)$, $T^2(n)$,... is too erratic to discover any useful non-trivial result, so the properties which should be investigated are those which relate to the sequence as a whole. A natural choice in the number of steps k until $T^k(n) = 1$ (assuming the conjecture). So:

Define: $f(n) = \inf\{m \mid T^m(n) = 1\}$.

For example, $f(1) = 0$, $f(2) = 1$, $f(3) = 7$, $f(4) = 2$, etc.; in general, $f(2n) = f(n) + 1$ and $f(2n-1) = f(6n-2) + 1$ $(n > 1)$. As we tabulate $f(n)$, certain patterns emerge.

An easy one is that for all $K \geq 1$, $f(8K + 4) = f(8K + 5)$.

To prove this, note that by operation of the algorithm:

$$\begin{array}{ccccccc} 8K+4 & \longrightarrow & 4K+2 & \longrightarrow & 2K+1 & \longrightarrow & 6K+4 \\ 8K+5 & \longrightarrow & 24K+16 & \longrightarrow & 12K+8 & \longrightarrow & 6K+4 \end{array}$$

and so $T^3(8K + 4) = T^3(8K + 5)$. So $f(T^3(8K + 4)) = f(T^3(8K + 5))$. But, by definition, $f(T^k(n)) = f(n) - k$, so $f(8K + 4) = f(8K + 5)$, as required.

Similarly $f(16K + 2) = f(16K + 3)$; $f(32K + 22) = f(32K + 23)$ etc.

In any interval, a few values of $f(n)$ will occur often. For instance, in the range 4900 to 4999, $f(n)$ takes only thirteen distinct values. In this range, $f(n) = 134$ for 29 values of n, $f(n) = 41$ has 72 solutions, and so on.

But the result I would like to investigate is more apparent by looking at differences between consecutive values of $f(n)$. So, with this in mind:

Define: $g(n) = f(n + 1) - f(n)$.

So, as a corollary to a previous result, $g(8K + 4) = 0$ $(K \geq 1)$.

For what follows next, we need a little elementary number theory.

Theorem. *If a, b are coprime, then there exist integers x, y such that $ax + by = 1$.* □

A rigorous proof is easy to find, but to construct x, y, we run the Euclidean algorithm in reverse; e.g., for $a = 12$, $b = 7$,

$$12 = 1 \times 7 + 5, \qquad 7 = 1 \times 5 + 2, \qquad 5 = 2 \times 2 + 1, \qquad 2 = 2 \times 1$$

Then $1 = 5 - 2 \times 2 = 5 - 2 \times (7 - 1 \times 5) = 3 \times 5 - 2 \times 7 = 3 \times (12 - 1 \times 7) - 2 \times 7 = 3 \times 12 - 5 \times 7$. So $x = 3$, $y = -5$ solves $12x + 7y = 1$.

An easy corollary is:

Theorem. *If d is a multiple of (a, b), the highest common factor of a and b, then there exist integers x, y such that $ax + by = d$. Furthermore, if x_0, y_0 are solutions to this, then so are $x_0 + kb$, $y_0 - ka$ for all integers k, and all solutions are of this form.* $\square$

If we now restrict ourselves to cases where a and b are coprime, we define

$$d_{a,b} = \min\{\sqrt{x^2 + y^2} \mid ax + by = n\}$$

which is well-defined, since the function $f(X) = \sqrt{(x_0 + bX)^2 + (y_0 - aX)^2}$ is convex. Now if X runs through the integers, $\min\{f(X) \mid X \in \mathbb{Z}\}$ is well-defined.

Now we return to the $3X + 1$ problem, and tabulate the values of $g(n)$ for $n = 1, 2, \ldots$, we may observe a rather curious result, namely that $d_{13,31}(g(n))$ is always "small"; in other words, the difference between two consecutive values of $f(n)$ is expressible as $13x + 31y$ for small values of x and y. For example, the values of $f(n)$ are tabulated below for $5385 \leq n \leq 5399$:

n	$f(n)$	$g(n)$	x	y	$[d(g(n))]^2$
5385	147	-80	1	-3	10
5386	67	80	-1	3	10
5387	147	-80	1	-3	10
5388	67	0	0	0	0
5389	67	-39	-3	0	9
5390	28	0	0	0	0
5391	28	88	2	2	8
5392	116	-49	1	-2	5
5393	67	93	0	3	9
5394	160	0	0	0	0
5395	160	-44	-1	-1	2
5396	116	0	0	0	0
5397	116	-49	1	-2	5
5398	67	0	0	0	0
5399	67	49	-1	2	5

where $d(k) = d_{13,31}(k)$.

This phenomenon occurs for all numbers greater than about 20, and as far as 20000 at least, and seems to hold even for random samples of twenty consecutive seven-digit numbers.

For smaller numbers, from 1 to 1000, computer checking has shown that the numbers 13 and 31 perform significantly better than any other pair with a, b coprime and less than 50. [2]

We consider the pair n, $n + 1$. The algorithm essentially consists of a halving or a trebling. If we look at the operation $X \mapsto 3X + 1$, we may neglect the addition of one, because it is fairly insignificant. Then for n to be transformed to 1, we need $a(n)$ halvings and $b(n)$ treblings, with $a(n) = b(n) = f(n)$. Similarly, for $n + 1$ to be transformed to 1, we need $a(n + 1)$ halvings and $b(n + 1)$ treblings with $a(n + 1) + b(n + 1) = f(n + 1)$.

So, to summarise,

$$n \cdot \frac{3^{b(n)}}{2^{a(n)}} \approx 1 \qquad a(n) + b(n) = f(n)$$
$$(n+1) \cdot \frac{3^{b(n+1)}}{2^{a(n+1)}} \approx 1 \qquad a(n+1) + b(n+1) = f(n+1)$$

For sufficiently large n,

$$\frac{3^{b(n)}}{2^{a(n)}} \approx \frac{3^{b(n+1)}}{2^{a(n+1)}}$$

Put $a = a(n + 1) - a(n)$, $b = b(n + 1) - b(n)$; then

$$\frac{3^b}{2^a} \approx 1 \tag{1}$$

Also, $a + b = a(n + 1) - a(n) + b(n + 1) - b(n) = f(n + 1) - f(n) = g(n)$.

$$a + b = g(n) \tag{2}$$
$$a, b \in \mathbb{Z} \tag{3}$$

Solving (1), (2) gives:

$$\begin{aligned} a &\approx g(n) \cdot \log_6 3 \\ b &\approx g(n) \cdot \log_6 2 \end{aligned}$$

Thus

$$\frac{b}{g(n)} \approx \log_6 2$$

But b, $g(n) \in \mathbb{Z}$, so to find possible values of $g(n)$, look for rational approximations to $\log_6 2$. To find good rational approximations, the obvious method is to look at the continued fraction expression. I give this in standard notation; explanations may be found in any number theory textbook.

$$\log_6 2 = [0, 2, 1, 1, 2, 2, 3, 1, 5, 2, \ldots]$$

giving successive convergents

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{5}{13}, \frac{12}{31}, \frac{41}{106}, \frac{53}{137}, \frac{306}{791}, \frac{665}{1719}, \ldots$$

The denominators of these fractions are the best possible values of $|g(n)|$. Thus, the best values for $|g(n)|$ are 1, 2, 3, 5, 13, 31, 106,..., and there are the numbers 13 and 31! Each successive pair $(1, 2)$; $(2, 3)$; $(3, 5)$;..., seems to be a "basis" for the values of $g(n)$ for a certain range of n (this occurs because of the simplifications involved in the analysis above).

(a, b)	n for which (a, b) is best basis
$(2, 3)$	$1 \leq n \leq 2$
$(3, 5)$	$3 \leq n \leq 8$
$(5, 13)$	$9 \leq n \leq 24$
$(13, 31)$	$25 \leq n \leq ?$

So, in fact, the result about 13 and 31 should not surprise us.

To end with, I will leave the interested reader (if there are any) with a few conjectures. I think these are all true, but very difficult to prove!

1. Let $h(n)$ be the highest member of the sequence n, $T(n)$, $T^2(n)$,.... It is clear that $\overline{\lim}(\frac{h(n)}{n}) = 1$, by taking $n = 2^k$, and that $\underline{\lim}(\frac{h(n)}{n}) = \infty$, by taking $n = 2^k - 1$; then $T^{2k}(n) = 3^k - 1$. Does $\lim(\frac{1}{N}\sum_1^N \frac{h(n)}{n})$ exist?
2. Given $k \in \mathbb{N}$, does there exist n for which $f(n) = f(n+1) = \cdots = f(n+k)$?
3. Set
$$m(n) = \begin{cases} 1, & \text{if } f(n) = f(n+1), \\ 0, & \text{otherwise} \end{cases}$$
Does $\lim(\frac{1}{n}\sum_1^N m(n))$ exist and if so, what is it?
4. Does $\lim(\frac{1}{N \log N}\sum_1^N f(n))$ tend to a finite non-zero limit?

The author would be delighted to hear of progress made on any of these problems!

References

[1] Gardner, M., *Wheels, Life and Other Mathematical Amusements*

[2] Many thanks to John Croft and Graham Nelson for these computer results.

This paper appeared in *Eureka*, the journal of the Archimedeans (the undergraduate mathematical society at the University of Cambridge), in 1989, when I was a second-year undergraduate. Looking back at it now (2005), I'm rather embarrassed by the poor quality of the writing. There are errors too; the second theorem is false if $(a, b) > 1$ – in this case, we can take k to be any rational with denominator (a, b). Of course, we only use the theorem for $(a, b) = (13, 31)$, so all the conclusions are fine.