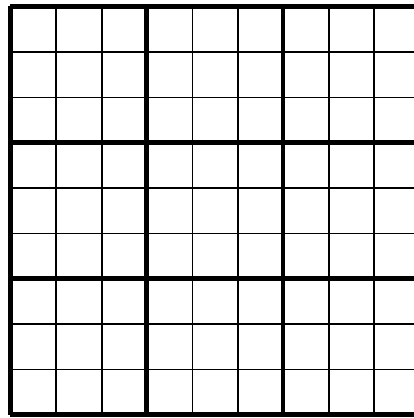# Mathematics of Sudoku II

Ed Russell
Frazer Jarvis*

February 15, 2006

Introduction

We recall that the idea of the Sudoku puzzle is extremely simple; the solver is faced with a 9×9 grid, divided into nine 3×3 blocks:

In some of these boxes, the setter puts some of the digits 1–9; the aim of the solver is to complete the grid by filling in a digit in every box in such a way that each row, each column, and each 3×3 box contains each of the digits 1–9 exactly once.

In an earlier article [1] of Bertram Felgenhauer and the second author, the number of different Sudoku grids was computed to be $N_0 = 6670903752021072936960 \approx 6.671 \times 10^{21}$. However, distinct solutions were treated as different, even when one could be transformed into another by some symmetry. However, you may feel that two grids should not be counted differently if, say, the second is just the first rotated by 90 degrees. Similarly, one can reflect a grid to get another valid grid – should these be counted differently? And one can relabel all the entries (exchanging 1s and 2s, for example) to give another grid. In this note, we want to explain how to refine the counting method of [1] to find how many *essentially different* grids there are, if we allow various possible symmetries.

Let us firstly study all the possible symmetries of a grid. That is, we want to find all the operations that we can make on a general Sudoku grid which preserve the property that every row, column and box contains each of the digits 1–9 exactly once.

As well as relabelling, reflecting and rotating, there are a few other operations one can perform. In order to define these, we need to define some terminology. We will say that a *block* will refer to one of the sub-3×3-squares within the 9×9 grid with boundary given by bold lines. A *stack* will consist of three blocks in a vertical 9×3 arrangement; a *band* will consist of three blocks in a horizontal 3×9 arrangement.

The complete list of operations that we can perform is:

1. Relabel (i.e., permute) the 9 digits;

2. Permute the three stacks;

---

*Contact address: Dr A.F.Jarvis, Department of Pure Mathematics, University of Sheffield, Sheffield S3 7RH, U.K., a.f.jarvis@shef.ac.uk
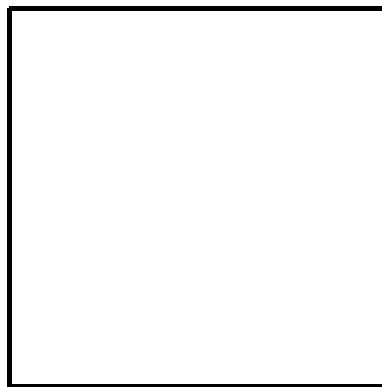
3. Permute the three bands;

4. Permute the three columns within a stack;

5. Permute the three rows within a band;

6. Any reflection or rotation.

In this article, we explain how to work out the number of grids which are "essentially different", where we regard two grids as essentially the same if there is a sequence of the operations (1)–(6) which transforms one of the grids into another.

We'll start by focusing on the operations (2)–(6), which are operations that are done on the grid squares, not on the *labels* within those squares. In a sense, we are going to think of these operations as *symmetries* of Sudoku puzzles, in the same way that a cube, for example, has symmetries – we can rotate a cube in several ways to put it back into the same position.

To describe the method, and the solution, we need to discuss some *group theory*.

Symmetries of geometrical objects form what is known as a *group*: this means that if we are given two symmetries, we can do one and then the other, and the result is itself a symmetry. For example, if we take a square frame of wire,

then we have eight symmetries:

- rot_0: rotation by 0 degrees [in other words, do nothing!];
- rot_1: rotate by 90 degrees clockwise;
- rot_2: rotate by 180 degrees;
- rot_3: rotate by 90 degrees anticlockwise;
- ref_0: reflect in the horizontal axis;
- ref_1: reflect in the top right–bottom left diagonal axis;
- ref_2: reflect in the vertical axis;
- ref_3: reflect in the top left–bottom right diagonal axis.

You might like to check that doing rot_1 and then ref_2, say, has exactly the same effect as doing ref_3. If we start with our square $\square$, we apply first rot_1 to get to $\mathrm{rot}_1(\square)$, and then apply ref_2 to that, to get $\mathrm{ref}_2(\mathrm{rot}_1(\square))$, this is the same as $\mathrm{ref}_3(\square)$. We just write $\mathrm{ref}_2\mathrm{rot}_1 = \mathrm{ref}_3$. Whichever pair of symmetries you take, doing one then the other will have the same effect as one of the eight symmetries on the list.

Formally, a group is a set G with a product rule such that:

1. if g_1, g_2 are both in G, then $g_1g_2 \in G$;

2. there is an element $e \in G$ such that $eg = ge = g$ for all $g \in G$ (so $e = \mathrm{rot}_0$ in the example above);

3. every element $g \in G$ has an inverse $g' \in G$, so that $gg' = g'g = e$;

4. $(g_1g_2)g_3 = g_1(g_2g_3)$ for any three elements g_1, g_2 and g_3 in G.

Here's a question that one might ask. Colour the edges of the square wire frame red and blue – how many essentially different ways are there to do this (where two colourings count the same if there is a symmetry turning one into the other)? It turns out that there is a mathematical theorem known as "Burnside's Lemma" which allows you to work out this sort of thing (in fact, although Burnside rediscovered it around 1900, it seems to have first been noticed by Cauchy around 1845!):

Write down all the possible symmetries, as we did above. For each symmetry, count the number of colourings which are fixed by the symmetry. Then the answer to the question is the average of these numbers.

Let's explain how this works in this example.

- The symmetry rot_0 is going to fix everything, of course; there are $2^4 = 16$ colourings of the square (each of the 4 edges can be coloured in 2 ways).

- The symmetry rot_1 will only fix 2 of the colourings, where each edge is red, or each edge is blue. (If a colouring is fixed by rotating it by 90 degrees, then each edge must be coloured the same way as its neighbour).

- The symmetry rot_2 will fix 4 of the colourings, the ones where the opposite edges are coloured in the same way. So the top and bottom edges are either both red or both blue, and the left and right edges are either both red or both blue.

- The symmetry rot_3 fixes 2 colourings, just as rot_1.

- The symmetry ref_0 fixes 8 colourings, those where the top and bottom edges are coloured the same.

- The symmetry ref_1 fixes 4 colourings, those where the top and right edges and the bottom and left edges are coloured the same.

- The symmetry ref_2 fixes 8 colourings, where the left and right edges are coloured the same.

- The symmetry ref_3 fixes 4 colourings, those where the top and left edges and the bottom and right edges are coloured the same.

Then the average of these is:

$$(16 + 2 + 4 + 2 + 8 + 4 + 8 + 4)/8 = 48/8 = 6,$$

and Burnside's Lemma tells us that this is exactly the answer to the original question of how many essentially different colourings there are.

We are trying to do the same to count the number of essentially different Sudoku grids. For simplicity, our group G will consist of all the symmetries built up from operations (2)–(6). These symmetries can be thought of as being operations permuting the 81 squares of the grid, and there is a computer program available, called GAP [2], which allows us to work with groups arising by permuting sets.

To take into account the operation (1) of relabelling, we are going to regard two grids as *equivalent* if one can be transformed into the other by relabelling. Then our symmetry group G is acting on grids; as in the example above, we want to count the number of grids which are fixed *up to equivalence* by a given symmetry, that is, the grids which are transformed by the symmetry into an equivalent grid.

For example, the following grid is equivalent to itself under a quarter turn:

1	2	4	5	6	7	8	9	3
3	7	8	2	9	4	5	1	6
6	5	9	8	3	1	7	4	2
9	8	7	1	2	3	4	6	5
2	3	1	4	5	6	9	7	8
5	4	6	7	8	9	3	2	1
8	6	3	9	7	2	1	5	4
4	9	5	6	1	8	2	3	7
7	1	2	3	4	5	6	8	9

where the relabelling takes $1 \to 3 \to 9 \to 7 \to 1$ and $2 \to 6 \to 8 \to 4 \to 2$, and 5 is fixed.

It turns out that the group G generated by the symmetries (2)–(6) above has 3359232 symmetries. To count the number of fixed points for all these would take an immense time. But just as the rotations by 90 degrees and by 270 degrees gave the same answer above, it turns out that there are 275 "conjugacy classes" (any 2 elements in the same class will have the same number of fixed grids), so if we choose one symmetry from each class, we only have to see what happens for these 275 symmetries – any symmetry will have the same number of fixed grids as one of these 275.

But for many symmetries, it is easy to see that there are no grids which are transformed to equivalent grids. For example, let's think about reflecting a Sudoku grid in a horizontal axis. Notice that all the numbers in the middle (fifth) row are unchanged. Since each of the nine digits occurs in this row, we see that there can be no relabelling; the grid must be transformed under the relection not just to an equivalent grid, but to *the same* grid. However, this is clearly impossible – this would mean, for example, that the two entries in the first column of rows 4 and 6 would have to be the same, but they are in the same box (B4), and two identical numbers in the same box is not allowed.

In fact, for all but 27 of these 275 symmetries, it can be shown that there are no grids at all transformed to equivalent grids. Using brute force methods like those in [1], the first author computed the number of elements transformed to equivalent grids by each of the remaining 27 symmetries. For example, it turns out that there are $9! \times 13056$ Sudoku grids which are equivalent to themselves under a quarter turn, like the example above, i.e., 13056 essentially different grids equivalent to themselves under a quarter turn.

Taking the average over all of the 3359232 group elements, we find that the number of essentially different Sudoku grids is 5472730538.

If we just allow relabelling, rotations, and permutations (but not reflections), the corresponding group H has order $6^8 = 1679616$ and has 484 conjugacy classes. Burnside's Lemma gives 10945437157 essentially different grids with these symmetries. (Both of these numbers have subsequently been confirmed by Kjell Fredrik Pettersen.)

Allowing only relabelling and rotations, so that the group just has 4 elements (rotation by 0, 90, 180 and 270 degrees), the same method gives 4595805644052864 essentially different grids. If we allow reflections as well, so that the symmetry group has 8 elements (the same as the symmetry group of the wire frame above), and it turns out that we get 2297902829591040 essentially different grids.

The programs and data are stored at `http://www.afjarvis.staff.shef.ac.uk/sudoku/`.

References

[1] B.Felgenhauer, F.Jarvis, *Mathematics of Sudoku I*, submitted to Mathematical Spectrum

[2] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4; 2005,* `http://www.gap-system.org`